

The Unified Approach for Best Choice Modeling Applied to Alternative-Choice Selection Problems

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Abstract

The objective of this paper is to show that the so-called *unified approach* to stopping problems with unknown cardinality introduced in Bruss (1984) proves to be efficient for solving other types of best-choice problems. We show that what we will call the *alternative-choice* stopping problem, which will be exemplified right away in Section 1, can be seen as a “two-sided” Secretary problem. This problem is instigated by a former problem of R. R. Weber (Cambridge University). Our approach yields for unknown cardinality the sharp lower bound $1/2$ for the probability of success. This problem is, at the same time, a special case of a model more generally based on $k \geq 2$ linearly ordered subsets. We shall also give the solution for such problems for k independent streams of arrivals. Our approach is elementary and self-contained.

Keywords. $1/e$ -law of best choice, optimal prediction, Weber stopping problem, secretary problem, selection of maximum elements, partially ordered set.

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1 Motivating Example and Theoretical Background

Consider an interviewer who interviews sequentially streams of candidates of men and women both arriving in $[0, 1]$. The total numbers of men and women are random variables and their arrival times are also i.i.d. The interviewer can rank in a unique way those of the same gender. At any time that a candidate presents itself the interviewer may stop and choose that candidate. Similarly to the well-known Secretary Problem, he is successful

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if he selects either the best of the men, or the best of the women. This is an example, with $k = 2$, of what we call an *alternative best-choice stopping problem* (which was motivated by Weber, 2012). We show that the unified approach to stopping problems (Bruss, 1984) can be applied to this problem. In the case of the men and women, and numbers of men and women are a priori unknown, we find that by choosing the first candidate to arrive after time $1/2$ who is best of its gender so far, the probability of success has the sharp lower bound of $1/2$.

Unified approach. An unknown number $N > 0$ of uniquely rankable options arrive on $[0, 1]$ with independent and identically distributed arrival times according to a continuous distribution function F defined on $[0, T]$. One speaks of a *success* if the decision maker accepts the absolute rank 1, or in other words, the very last relatively best (record). Let σ_t be the strategy to wait until time t and then to select the first record (relative best) option thereafter, if any (the stopping time induced by σ_t is set equal to T , otherwise). We first recall the $1/e$ -law of Bruss (1984). Let

$$t^* = \inf\{t : F(t) = 1/e\} =: F^{-1}(1/e). \quad (1)$$

Then, the $1/e$ -law says: First, the strategy to stop on the first record appearing at time t^* or later, σ_{t^*} , say, succeeds with probability at least $1/e$. Second, conditioned on $\{N = n\}$ the bound $1/e$ is the limit of decreasing success probabilities as $n \rightarrow \infty$. Third, the probability of σ_{t^*} (called $1/e$ -strategy) selecting no candidate at all is exactly $1/e$. It is for this *triple* coincidence of the number $1/e$ that the author called the result “ $1/e$ -law”. This result came as a surprise since, in general, a success probability near $1/e$ seemed at that time out of reach. See the comments of Samuels (1985) in *Mathematical Reviews*.

Alternative choice and related work

What we will call the *problem of alternative choice* is the continuous-time model of Bruss (1984) but now with $k \geq 1$ classes of options. For example the problem of choosing with one choice either the best male, or the best female candidate. Observed options are supposed to be uniquely rankable among all options in the same class but typically not comparable with options from another class.

This version of Weber’s problem as well as the underlying work may be seen as a selection model based on partially ordered sets (posets), or, as we will argue, also as a multicriteria selection model. There are several

interesting articles which are related with our work. Selection problems involving posets seem to have been first studied by Stadjé (1980). Morayne et al. (2008) propose a novel interesting universal algorithm for posets where the cardinality of the set is known. The ordering structure of the poset need not be known; in particular, the number of maximal elements in the set need not be known.

Freij and Wästlund (2010) present a generalized strategy in continuous time for a similar model to theirs, where the size of the poset is not fixed. This is partially in the spirit of the underlying paper following the unified approach. Their work can be seen as a generalization of the previous work. Interestingly, they also obtain the ubiquitous $1/e$ lower bound for the success probability in quite a general setting.

Kumar et al. (2011) and Garrod and Morris (2013) studied independently a similar stopping problem on a poset of n elements where both n and the number k of maximal elements are known. In that case, the probability of success of the strategy they propose can be improved to $k^{-1/(k-1)}$, a value which is bound to turn up in our generalization by the unified approach.

Gnedin’s multicriteria models (Gnedin, 1981) share some similarity with our model. As he explained: it is unlikely that a secretary is best in several criteria at the same time; the probability that this happens tends to 0 as the number n of applicants tends to infinity. That is why the model is similar to one where each secretary belongs to only one category.

In selection problems with more information on the number of arrivals of options, so-called quasi-stationarity is a very desirable property for obtaining closed-form solutions. The approach of Bruss and Samuels (1990) can be adapted for options stemming from different classes, but here again the unified approach without any knowledge at all is not that much weaker in the result and therefore, taking things together, clearly preferable for applications.

Parts of the precedingly cited articles go deeper into certain directions not mentioned here. In particular our result is partially intrinsic in the interesting paper of Garrod and Morris (2013). However, first, the present article shows that the unified approach on which we focus yields the results in an elementary and self-contained way, and has, in particular for the case $k = 2$ much appeal for applications in real life. This is seen in several quite different examples in Bruss and Ferguson (2002) for the case $k = 1$. Second, according to their list of references, none of the authors on poset selection problems cited above has referenced the unified approach of 1984 which adds to our motivation for this article.

2 Main results

Threshold-time strategy. Similarly to what has been done in the $1/e$ -law, we aim to find a time threshold t^* after which we decide to stop on the first encountered record, regardless of the class to which this record belongs. Let σ_t be the strategy to wait until time t and then to select the first record option thereafter, if any; the stopping time induced by σ_t is set equal to T otherwise, meaning a complete failure (no choice at all).

Model. Let $N_1, N_2, \dots, N_k \in \{1, 2, \dots\}$ with no other information available on these numbers than being positive integers. They are seen as the numbers of options of class 1, respectively class 2, \dots , and so on. We consider iid arrival times $X_{l,i}$ ($1 \leq i \leq N_k$, $1 \leq l \leq k$) of the options on $[0, 1]$ so that the r th option of type j arrives at time $X_{j,r}$. Since there are k non-empty classes, there are also k options which are best in their class by definition, because we supposed that they can be ranked in a unique way. These will be called “maxima”. We would like to stop online on any of the maxima.

The $1/2$ -rule

In the view of real life applications, the case $k = 2$ is particularly appealing. First, the decision rule will be easy to remember. Second, it also covers the problem predicting the best or alternatively the worst outcome from a stream of events. Indeed, we can construct the two streams artificially as follows. Consider one stream of rankable secretaries and remember the very first secretary. Then, all the subsequent secretaries with a better rank than the remembered secretary belong to “class 1” and the secretaries with a worse rank will belong to class 2. Hence this serves the problem of finding either the best or the worst of all. We also mention here that Bayon et al. (2016) have studied different cases of selecting the best or worst secretary from a sequence of secretaries with random length.

If one single stream of options (not necessarily secretaries) can be divided, on some rational, in two complementary classes (“positive” and “negative”), then this construction applies of course as well. In the case of a financial index, positive increments belong in one class, negative increments belong to the second class.

Theorem 1 is Theorem 2 with $k = 2$, but we state Theorem 1 first because of its appealing simplicity.

Theorem 1. *In the model described above in Section 2, if there are two non-empty classes of options, then stopping time $\sigma_{1/2}$ which stops on the*

first record belonging to either of the two classes after time $1/2$ has success probability of at least $1/2$. (This holds thus for any number of options as long as both classes are non-empty.)

The proof of Theorem 1 is immediate from the proof following Theorem 2 and can therefore be postponed.

Remark 1. Here again, this is the best fixed-threshold strategy yielding the success probability $1/2$ for such strategies. It cannot be improved because the probability of success of $\sigma_{1/2}$ converges to $1/2$ if the numbers of options in both classes tend to infinity. This confirms the result in the discrete setting from Garrod and Morris (2013).

The general case

Theorem 2. Let $N_1, N_2, \dots, N_k \geq 1$ denote the unknown number of options of class 1, respectively class 2, \dots , respectively class k . We assume that the arrival times of all options are independent and that their distribution is uniform on $[0, 1]$. We suppose that inside each class, the options are uniquely rankable against one another.

Then, the strategy σ_{t_k} which stops on the first record (=relative maximum in its class) appearing after time $t_k := k^{-1/(k-1)}$ satisfies

- (a) σ_{t_k} succeeds with probability at least t_k .
- (b) Seen as a function of the unknowns $N_1 = n_1, N_2 = n_2, \dots, N_k = n_k$, the success probability decreases monotonically in all n_j with limit t_k as $\min_{1 \leq j \leq k} n_j \rightarrow \infty$.

Proof. (a) Define $p(t)$ as the probability that the strategy σ_t will turn out (at time 1) to be a success. (Remark 2 is more specific about issues related to this “probability”.) Denote by T_1, T_2, \dots, T_k the arrival times of the maxima of class 1, respectively class 2, \dots , respectively class k . Consider the events

$$E_i(t) := \{T_1, \dots, T_{k-i} < t\} \cap \{T_{k-i+1}, \dots, T_{k-1}, T_k \geq t\}, \text{ for } i = 1, 2, \dots, k, \quad (2)$$

and let

$$S_i(t) = E_i(t) \cap \{\sigma_t \text{ turns out to be a success}\}. \quad (3)$$

We have $p(t) = \sum_{i=1}^k \binom{k}{i} P(S_i(t))$, because there are $\binom{k}{k-i}$ ways to choose $k-i$ i.i.d. random arrival times out of a set of k arrival times. Since the classes are non-empty, this is independent of their size since a maximum in each class must exist.

Let $T_{(i)} := \min(T_{k-i+1}, \dots, T_k)$ be the arrival time of the first maxima, restricted to the last i classes, arriving in $[t, 1]$. We notice that when $E_i(t)$ happens, the strategy leads to a success if stopping occurs at time $T_{(i)}$, by definition of σ_t . Because the T_j 's are uniformly distributed, the conditional density of $T_{(i)}$ given that $E_i(t)$ happens, denoted by $f^{(i)}$, is

$$f^{(i)}(s) = \chi_{[0,1]}(s) i(1-s)^{(i-1)}(1-t)^{-i}, \quad s \in \mathbb{R}, \quad (4)$$

where χ_A denotes the indicator function of a set A . We may now write

$$\begin{aligned} p(t) &= \sum_{i=1}^k \binom{k}{i} P(\sigma_t \text{ succeeds} | E_i(t)) P(E_i(t)) \\ &= \sum_{i=1}^k \binom{k}{i} \left(\int_t^1 f^{(i)}(s) P(\sigma_t \text{ succeeds} | T_{(i)} = s, E_i(t)) ds \right) t^{k-i} (1-t)^i \\ &= \sum_{i=1}^k \binom{k}{i} \left(\int_t^1 f^{(i)}(s) P(A(t, s) | T_{(i)} = s, E_i(t)) ds \right) t^{k-i} (1-t)^i, \end{aligned} \quad (5)$$

where $A(t, s)$ is defined as the event that no record of any class is seen on the interval $[t, s]$, for $0 \leq t < s < 1$. We now use

$$P(A(t, s) | T_{(i)} = s, E_i(t)) \geq (t/s)^i. \quad (6)$$

To keep the current proof streamlined, we will prove (6) in a technical lemma after the current proof. Using inequality (6) in (5), we obtain

$$p(t) \geq \int_t^1 \sum_{i=1}^k \binom{k}{i} t^{k-i} (1-t)^i i (1-s)^{(i-1)} (1-t)^{-i} (t/s)^i ds. \quad (7)$$

Cancelling out a few terms, Equation (7) simplifies to

$$p(t) \geq \int_t^1 t^k \sum_{i=1}^k \binom{k}{i} i (1-s)^{(i-1)} s^{-i} ds. \quad (8)$$

We can multiply and divide each term of the sum in (8) by $(1-s)s^k$, so as to obtain

$$p(t) \geq \int_t^1 \frac{t^k}{(1-s)s^k} \sum_{i=1}^k \binom{k}{i} i (1-s)^i s^{k-i} ds, \quad (9)$$

and the sum can be interpreted as the expectation of a random variable with distribution $\text{Binomial}(k, 1-s)$. Therefore the value of the sum $k(1-s)$. Finally,

$$p(t) \geq \int_t^1 t^k k s^{-k} ds = \frac{k}{k-1} (t - t^k) =: h_k(t), \quad (10)$$

The function $h_k(t)$ is maximized in $t = t_k$, hence $p(t_k) \geq h_k(t_k) = t_k$.

(b) The proof of (a) is organized in such a way that we can argue here without further calculations. Recall now (5) and . Note that conditioned on $N_j = n_j \geq n$, the event that of at least one arrival in class j has an increasing probability as n_j increases, for all j . But then the conditional probabilities of $A(t, s)$ in the integrands must be non-increasing and hence converge to the shown lower bounds $(t/s)^j$. Since this holds for all $0 < t \leq s < 1$ pointwise in s , it must hold for the corresponding integrals. As they all have non-negative integrands, it must hold for their sum. \square

Lemma 1. *Suppose that $n \geq 0$ random variables are independent and uniformly distributed on $[0, 1]$. These random variables represent the arrival times of n uniquely rankable options. The probability that no relative maximum is seen in the interval $[t, s]$ is greater or equal to t/s .*

Proof. We first prove the result for one class of options. The following argument can already be found in Bruss and Yor (2012) and in a slightly different form in Freij and Wästlund (2010). Look at the following constellation of points on $[0, 1]$.



There are two possibilities. Either there is no arrival at all in $[0, s]$, in which case $A(t, s)$ happens automatically, or there is at least one. But if there is at least one, then there is a best among them. Since a conditional uniform random variable stays uniform on the given interval, any point has probability t/s of arriving in $[0, t]$, given it arrives in $[0, s]$. Hence, since $t, s \leq 1$, we have that the probability of no record (in class 1) in $[t, s]$ is either t/s or one, that is at least t/s .

Now, more generally, in the context of Theorem 2, this argument can be applied to any class of options. If we consider i classes, by independence of the arrival times, the probability of the event $A(t, s)$ that no relative maximum (record) from these i classes is seen in $[t, s]$ is then simply at least $(t/s)^i$. \square

Remark 2. Remember the definition of $p(t)$ at the beginning of the proof of Theorem 2. Note that the event of the last record at time t in its class is not measurable in the filtration generated by the observations up to time t . This probability would only be defined conditionally. The problem of giving a lower bound is however not affected by this.

Remark 3. In the context of Theorem 2, the probability of not stopping at all is $k^{-1}t_k$. Hence for $k > 1$ we cannot have another “triple” coincidence of the three crucial values as in the $1/e$ -law for $k = 1$.

Open problem. Notice that Theorem 2 does not mention the word optimality. Optimality is an open problem, as it also still is for the $1/e$ -law, as reconfirmed in Bruss and Yor (2012, Section 6.4). The latter have shown optimality of their solution for the so-called last-arrival problem which looks very similar, so that the conjecture of optimality is now almost compelling. However, the complete lack of information about the number of options is in record-based problems even more delicate.

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